Solitons on a zigzag-runged ladder lattice

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The nonlinear dynamical model on a two-leg ladder lattice with the rungs arranged into a zigzag chain is proposed. The lattice contains two structure elements (molecules) in the unit cell. As a result, the system exhibits the two-branch spectrum in its low-amplitude limit. The similar two branches are shown to be observed in high-amplitude soliton solutions too. The integrability of the model is proved and the one-soliton solutions are explicitly presented and analyzed. The model Hamiltonian, as well as the basic conserved quantities are found.

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The integrable one-component [1] and multicomponent [2,3] nonlinear Schrödinger systems on one-dimensional lattices arise as appropriate discretizations of their respective continuous counterparts, namely the Zakharov-Shabat [4] and Manakov [5] ones. However, the power of any discretized model appears to be more pronounced as compared with its continuous ancestor. Thus, using the integrable Ablowitz-Ladik model [1] we have managed to reveal the localized solitonic modes in discrete Davydov-Kyslukha nonlinear system [6,7] as well as in a standard discrete nonlinear Schrödinger one [7]. The physical origin of these oscillations turns out to be the same as that of intrinsic localized modes observable in a pure unharmonic lattice [8,9] in view of the close relationship between the nonlinear Schrödinger and the nonlinear mechanical lattice systems via rotating-wave approximation [10,11]. The multicomponent integrable nonlinear models [2,3] may in principle be applied to the investigation of even more sophisticated dynamical systems.

Recently, we have developed the multicomponent integrable nonlinear model on a multileg ladder lattice [12,13]. It has permitted us to describe the slalom soliton dynamics on a ladder lattice with zigzag distributed on-site impurities [14,15] and has furnished insights into the nature of an attractive-repulsive alternative in an effective soliton interaction with the modified transverse bond [14].

Another approach dealing mainly with the soliton localization on a double-chain (ladder) lattice has been demonstrated within the framework of two coupled one-dimensional Ablowitz-Ladik equations [16]. The similar idea to couple two known integrable discrete nonlinear systems, namely Ablowitz-Ladik [1] and Toda [17] systems, has actually been explored to investigate the solitonic energy transfer in a coupled exciton-vibron system [18].

It is interesting to note that ladder lattice structures are now very popular objects also for the experimental detection [19,20] and theoretical description [20] of spatially localized excitations (breathers) in Josephson-junction arrays.

Finally, the nonlinear mechanical model of the Fermi-Pasta-Ulam-type has recently been invoked for the experimental and theoretical investigation of pulselike deformations propagating through the discrete geophysical medium with the Hertz contact interaction between the structure elements [21].

Thus, we clearly see that the discrete nonlinear models play a fundamental role in a variety of physical problems. The integrable models on ladder lattices among them appear to be the most perspective ones due in part to their integrability and realistic quasi-one-dimensional character of the primary lattice structure.

In the present report we study the integrable nonlinear model on a two-leg ladder lattice with the rungs arranged into a zigzaglike chain. Here, we would like to stress that such a lattice has nothing to do with those considered in our previous publications [12–15]. Consequently, the model of interest is suggested to be another one.

We start with the explicit presentation of our model

$$\begin{aligned}
&\frac{+i\dot{q}_-(n)+2\omega_0 q_-(n)}{1+q_-(n)r_-(n)}+\omega_t^- q_+(n)+\omega_t^+ q_+(n-1)\\
&\quad+\omega_l^+ q_-(n-1)[1+q_+(n-1)r_+(n-1)]\\
&\quad+\omega_l^- q_-(n+1)[1+q_+(n)r_+(n)]+\omega_l^+ q_+(n-1)\\
&\quad\times[q_+(n-1)r_-(n)+q_-(n)r_+(n)]+\omega_l^- q_+(n)\\
&\quad\times[q_+(n)r_-(n)+q_-(n)r_+(n-1)]=0
\end{aligned}\tag{1}$$

$$\begin{aligned}
&\frac{-i\dot{r}_-(n)+2\omega_0 r_-(n)}{1+r_-(n)q_-(n)}+\omega_t^+ r_+(n)+\omega_t^- r_+(n-1)\\
&\quad+\omega_l^- r_-(n-1)[1+r_+(n-1)q_+(n-1)]\\
&\quad+\omega_l^+ r_-(n+1)[1+r_+(n)q_+(n)]+\omega_l^- r_+(n-1)\\
&\quad\times[r_+(n-1)q_-(n)+r_-(n)q_+(n)]+\omega_l^+ r_+(n)\\
&\quad\times[r_+(n)q_-(n)+r_-(n)q_+(n-1)]=0
\end{aligned}\tag{2}$$

$$\begin{aligned}
&\frac{+i\dot{q}_+(n)+2\omega_0 q_+(n)}{1+q_+(n)r_+(n)}+\omega_t^+ q_-(n)+\omega_t^- q_-(n+1)\\
&\quad+\omega_l^- q_+(n+1)[1+q_-(n+1)r_-(n+1)]\\
&\quad+\omega_l^+ q_+(n-1)[1+q_-(n)r_-(n)]+\omega_l^- q_-(n+1)\\
&\quad\times[q_-(n+1)r_+(n)+q_+(n)r_-(n)]+\omega_l^+ q_-(n)\\
&\quad\times[q_-(n)r_+(n)+q_+(n)r_-(n+1)]=0
\end{aligned}\tag{3}$$

$$\frac{-i\dot{r}_+(n)+2\omega_0 r_+(n)}{1+r_+(n)q_+(n)}+\omega_t^- r_-(n)+\omega_t^+ r_-(n+1)$$
$$+\omega_l^+ r_+(n+1)[1+r_-(n+1)q_-(n+1)]$$
$$+\omega_l^- r_+(n-1)[1+r_-(n)q_-(n)]+\omega_l^+ r_-(n+1)$$
$$\times[r_-(n+1)q_+(n)+r_+(n)q_-(n)]+\omega_l^- r_-(n)$$
$$\times[r_-(n)q_+(n)+r_+(n)q_-(n+1)]=0, \qquad (4)$$

as a sufficient object to give some basic definitions and as a most natural tool to trace the structure of intersite linear couplings (bonds), and hence, to imagine more clearly the primary structure of the whole lattice. Namely, we can safely prescribe the dynamical variables $q_-(n), r_-(n)$ and $q_+(n)$, $r_+(n)$ to be the field amplitudes (or amplitudes of intramolecular excitations) associated with the lattice sites, respectively, on left $(-)$ and right $(+)$ straight chains (legs) of the lattice within nth unit cell. Then the quantities ω_l^-, ω_l^+ and ω_t^-, ω_t^+ are seen to characterize the strength of longitudinal (l) and transverse (t) intersite linear couplings, respectively, regardless of their possible time dependences. The overdot in Eqs. (1)–(4) stands for the derivative with respect to dimensionless time τ, whereas the longitudinal numerical coordinate n is assumed to run from minus to plus infinity. Finally, the terms proportional to ω_0 describe the regular energy shift and could be easily eliminated by the standard gauge transformation of field amplitudes.

It is easy to conclude that every site of adopted two-leg ladder lattice is linearly coupled to four of its neighbors (to two on the same leg of the ladder and to two on the opposite leg) in contrast to the two-leg version of the already known integrable model [12–14], where only three neighbors in linear couplings are involved.

In general, the coupling parameters ω_l^-, ω_l^+ and ω_t^-, ω_t^+ are proved to be arbitrary functions of time. The freedom in choosing their particular time modulations appears to provide a practically inexhaustible source of parametrically driven physical systems integrable by the inverse scattering transform.

It is worthwhile to stress that the nonlinear model of interest may be readily written in rather compact Hamiltonian form

$$+i\dot{q}_\mp(n)=[1+q_\mp(n)r_\mp(n)]\partial H/\partial r_\mp(n), \qquad (5)$$

$$-i\dot{r}_\mp(n)=[1+q_\mp(n)r_\mp(n)]\partial H/\partial q_\mp(n). \qquad (6)$$

Here, the model Hamiltonian

$$H=-\omega_l^- I_l^- -\omega_t^- I_t^- -2\omega_0 I_0 -\omega_t^+ I_t^+ -\omega_l^+ I_l^+, \qquad (7)$$

is totally determined by the conserved quantities

$$I_l^-=\sum_{m=-\infty}^{\infty} q_+(m)r_+(m-1)[1+q_-(m)r_-(m)]$$
$$+\sum_{m=-\infty}^{\infty} q_-(m+1)r_-(m)[1+q_+(m)r_+(m)]$$
$$+\sum_{m=-\infty}^{\infty}\frac{1}{2}[q_+^2(m)r_-^2(m)+q_-^2(m)r_+^2(m-1)], \qquad (8)$$

$$I_t^-=\sum_{m=-\infty}^{\infty}[q_+(m)r_-(m)+q_-(m)r_+(m-1)], \qquad (9)$$

$$I_0=\sum_{m=-\infty}^{\infty}\ln\{[1+q_-(m)r_-(m)][1+q_+(m)r_+(m)]\}, \qquad (10)$$

$$I_t^+=\sum_{m=-\infty}^{\infty}[q_-(m)r_+(m)+q_+(m)r_-(m+1)], \qquad (11)$$

$$I_l^+=\sum_{m=-\infty}^{\infty} q_-(m)r_-(m+1)[1+q_+(m)r_+(m)]$$
$$+\sum_{m=-\infty}^{\infty} q_+(m-1)r_+(m)[1+q_-(m)r_-(m)]$$
$$+\sum_{m=-\infty}^{\infty}\frac{1}{2}[q_-^2(m)r_+^2(m)+q_+^2(m)r_-^2(m+1)], \qquad (12)$$

via an appropriate multiplication by the coupling parameters.

Although the Poisson brackets related to the model (1)–(4) are turned out to be nonstandard, they are unable to cause any discrepancy in physical applications. Indeed, introducing the corrected amplitudes

$$Q_\mp(n)=\sqrt{[q_\mp(n)/r_\mp(n)]\ln[1+q_\mp(n)r_\mp(n)]}, \qquad (13)$$

$$R_\mp(n)=\sqrt{[r_\mp(n)/q_\mp(n)]\ln[1+q_\mp(n)r_\mp(n)]}, \qquad (14)$$

we might always convert the original model (1)–(4) into the standard form

$$+i\dot{Q}_\mp(n)=\partial H/\partial R_\mp(n), \qquad (15)$$

$$-i\dot{R}_\mp(n)=\partial H/\partial Q_\mp(n). \qquad (16)$$

Here, of course, H must be written in terms of the amplitudes $Q_-(n), R_-(n)$ and $Q_+(n), R_+(n)$.

Remarkably, the corrected model (15),(16) possesses the same linear part as the original one (1)–(4) and hence, exhibits the same low-amplitude spectrum.

The model (1)–(4) is integrable by the method of inverse scattering transform insofar as it actually has been decoded from the Lax equation

$$\dot{L}(n|z)=A(n+1|z)L(n|z)-L(n|z)A(n|z), \qquad (17)$$

with the following spectral operator:

$$L(n|z)=\begin{pmatrix} z^2-q_+(n)r_-(n), & izq_-(n)+iz^{-1}q_+(n) \\ izr_+(n)+iz^{-1}r_-(n), & z^{-2}-r_+(n)q_-(n) \end{pmatrix}. \tag{18}$$

The corresponding evolution operator $A(n|z)$ is determined from the Lax equation (17) under the assumption that it must be expanded in the same powers of spectral parameter z as $L^2(n|z)$, i.e., its matrix elements $A_{jk}(n|z)$ must be sought in the form

$$A_{11}(n|z)=a_{11}(n)z^4+b_{11}(n)z^2+c_{11}(n)+d_{11}(n)z^{-2}, \tag{19}$$

$$A_{12}(n|z)=a_{12}(n)z^3+b_{12}(n)z+c_{12}(n)z^{-1}+d_{12}(n)z^{-3}, \tag{20}$$

$$A_{21}(n|z)=a_{21}(n)z^3+b_{21}(n)z+c_{21}(n)z^{-1}+d_{21}(n)z^{-3}, \tag{21}$$

$$A_{22}(n|z)=a_{22}(n)z^2+b_{22}(n)+c_{22}(n)z^{-2}+d_{22}(n)z^{-4}. \tag{22}$$

Specifically, for the functions $a_{jk}(n)$, $b_{jk}(n)$, $c_{jk}(n)$, and $d_{jk}(n)$, we have

$$a_{11}(n)=+i\omega_l^-, \tag{23}$$

$$b_{11}(n)=+i\omega_t^-+i\omega_l^-q_-(n)r_+(n-1), \tag{24}$$

$$\begin{aligned} c_{11}(n)=&+i\omega_0+i\omega_t^-q_-(n)r_+(n-1)+i\omega_l^-q_+(n)r_+(n-1)\\ &\times[1+q_-(n)r_-(n)]+i\omega_l^-q_-(n)r_-(n-1)\\ &\times[1+q_+(n-1)r_+(n-1)]+i\omega_l^-q_-^2(n)r_+^2(n-1), \end{aligned} \tag{25}$$

$$d_{11}(n)=+i\omega_l^+q_+(n-1)r_-(n), \tag{26}$$

$$a_{12}(n)=-\omega_l^-q_-(n), \tag{27}$$

$$\begin{aligned} b_{12}(n)=&-\omega_t^-q_-(n)-\omega_l^-q_+(n)[1+q_-(n)r_-(n)]\\ &-\omega_l^-q_-^2(n)r_+(n-1), \end{aligned} \tag{28}$$

$$\begin{aligned} c_{12}(n)=&+\omega_t^+q_+(n-1)+\omega_l^+q_-(n-1)\\ &\times[1+q_+(n-1)r_+(n-1)]+\omega_l^+q_+^2(n-1)r_-(n), \end{aligned} \tag{29}$$

$$d_{12}(n)=+\omega_l^+q_+(n-1), \tag{30}$$

$$a_{21}(n)=-\omega_l^-r_+(n-1), \tag{31}$$

$$\begin{aligned} b_{21}(n)=&-\omega_t^-r_+(n-1)-\omega_l^-r_-(n-1)\\ &\times[1+r_+(n-1)q_+(n-1)]\\ &-\omega_l^-r_+^2(n-1)q_-(n), \end{aligned} \tag{32}$$

$$\begin{aligned} c_{21}(n)=&+\omega_t^+r_-(n)+\omega_l^+r_+(n)[1+r_-(n)q_-(n)]\\ &+\omega_l^+r_-^2(n)q_+(n-1), \end{aligned} \tag{33}$$

$$d_{21}(n)=+\omega_l^+r_-(n), \tag{34}$$

$$a_{22}(n)=-i\omega_l^-r_+(n-1)q_-(n), \tag{35}$$

$$\begin{aligned} b_{22}(n)=&-i\omega_0-i\omega_t^+r_-(n)q_+(n-1)-i\omega_l^+r_+(n)\\ &\times q_+(n-1)[1+r_-(n)q_-(n)]-\omega_l^+r_-(n)\\ &\times q_-(n-1)[1+r_+(n-1)q_+(n-1)]\\ &-\omega_l^+r_-^2(n)q_+^2(n-1), \end{aligned} \tag{36}$$

$$c_{22}(n)=-i\omega_t^+-i\omega_l^+r_-(n)q_+(n-1), \tag{37}$$

$$d_{22}(n)=-i\omega_l^+. \tag{38}$$

The results of the previous paragraph when combined with the auxiliary linear problems

$$\mathbf{u}(n+1|z)=L(n|z)\mathbf{u}(n|z), \tag{39}$$

$$\dot{\mathbf{u}}(n|z)=A(n|z)\mathbf{u}(n|z), \tag{40}$$

(where $\mathbf{u}(n|z)$ is the two-component column vector) allow us to integrate the models of interest (1)–(4). However, the detailed description of the whole integration machinery goes beyond the scope of the present report. Instead, we will discuss the simplest one-soliton solutions of the models although being obtained within the framework of inverse scattering transform.

We restrict ourselves to the case of reduction $r_-(n)=q_-^*(n)$, $r_+(n)=q_+^*(n)$ corresponding to the attractive type of nonlinearity and bright soliton solutions. Then the dynamical Eqs. (1)–(4) will be mutually consistent under constraint $H^*=H$. As a consequence, the parameter ω_0 must be purely real, while ω_l^-, ω_l^+, and ω_t^-, ω_t^+ can be parametrized as follows

$$\omega_l^{\pm}=\omega_l\exp(\pm i\gamma_l), \tag{41}$$

$$\omega_t^{\pm}=\omega_t\exp(\pm i\gamma_t), \tag{42}$$

where the parameters ω_l,γ_l and ω_t,γ_t are supposed to be the real ones. For the sake of simplicity, all of these parameters will be taken to be time independent.

It is interesting to note that analogous to the low-amplitude (linear) modes, the one-soliton solutions of our nonlinear models (1)–(4) can be separated into two ($\nu=0$ and $\nu=1$) one-soliton branches

$$q_-^{(\nu)}(n)=\frac{\sinh(\mu)\exp\left[+i\kappa n+i\left(\theta-\frac{\kappa}{4}-\frac{\pi}{2}\nu\right)-i\tau\Omega_\nu(\kappa|\mu)\right]}{\cosh\left\{2\mu\left[n-\frac{1}{4}-x(0)-\tau v_\nu(\kappa|\mu)\right]\right\}}, \tag{43}$$

$$q_+^{(\nu)}(n)=\frac{\sinh(\mu)\exp\left[+i\kappa n+i\left(\theta+\frac{\kappa}{4}+\frac{\pi}{2}\nu\right)-i\tau\Omega_\nu(\kappa|\mu)\right]}{\cosh\left\{2\mu\left[n+\frac{1}{4}-x(0)-\tau v_\nu(\kappa|\mu)\right]\right\}}, \tag{44}$$

characterized by two different ($\nu=0$ and $\nu=1$) cyclic frequencies

$$\Omega_\nu(\kappa|\mu) = -2\omega_0 - 2\omega_l \cosh(2\mu)\cos(\kappa-\gamma_l) - (-1)^\nu 2\omega_t \cosh(\mu)\cos\left(\frac{\kappa}{2}-\gamma_t\right), \quad (45)$$

and by two different ($\nu=0$ and $\nu=1$) typical longitudinal velocities

$$v_\nu(\kappa|\mu) = \frac{\omega_l}{\mu}\sinh(2\mu)\sin(\kappa-\gamma_l) + (-1)^\nu \frac{\omega_t}{\mu}\sinh(\mu) \times \sin\left(\frac{\kappa}{2}-\gamma_t\right). \quad (46)$$

Here, we suppose the wave number κ to be bounded to the interval $-\pi+2\gamma_t \leqslant \kappa \leqslant +\pi+2\gamma_t$. In the limit of extremely long solitons $\mu\to 0$, the frequencies $\Omega_\nu(\kappa|\mu)$ approach to the frequencies of linear spectrum $\Omega_\nu(\kappa|0)$ ($\nu=0,1$). In the same limit $\mu\to 0$, the quantities $v_\nu(\kappa|\mu)$ acquire the meaning of group velocities of respective planar waves

$$v_\nu(\kappa|0) \equiv \partial\Omega_\nu(\kappa|0)/\partial\kappa, \quad (\nu=0,1). \quad (47)$$

The precise meaning of each velocity $v_\nu(\kappa|\mu)$ at an arbitrary longitudinal size of soliton $d\sim\coth 2\mu$ may be understood calculating the mean longitudinal coordinate of the respective soliton wave packet on corrected [according to Eqs. (13),(14)] one-soliton amplitudes

$$x_\nu \equiv \frac{\sum_{n=-\infty}^{\infty}[(n-1/4)Q_-^{(\nu)}(n)R_-^{(\nu)}(n) + (n+1/4)Q_+^{(\nu)}(n)R_+^{(\nu)}(n)]}{\sum_{n=-\infty}^{\infty}[Q_-^{(\nu)}(n)R_-^{(\nu)}(n) + Q_+^{(\nu)}(n)R_+^{(\nu)}(n)]}. \quad (48)$$

We obtain

$$x_\nu = x(0) + \tau v_\nu(\kappa|\mu). \quad (49)$$

Hence, $v_\nu(\kappa|\mu)$ is nothing but the longitudinal velocity of the corrected one-soliton pattern belonging to νth one-soliton branch, while $x(0)$ is the initial mean longitudinal coordinate of this pattern.

The quantity, written in the denominator of definition (48) determines the number of excitations in the one-soliton solution of νth branch. It owes to be time independent thanks to the conserved quantity I_0 [see Eq. (10) taken in terms of corrected amplitudes (13),(14)]. The direct calculations confirm this statement yielding for the number of excitations the same time-independent value 2μ irrespective of a particular one-solitonic branch.

Summarizing, we have developed the nonlinear dynamical model on a zigzag-runged ladder lattice integrable by the inverse scattering transform. We have found its Hamiltonian formulation with the Hamiltonian function determined by the superposition of basic conserved quantities and describe how to rewrite the model in terms of physically corrected amplitudes. We have proved the model integrability and presented its one-soliton solutions, which happen to manifest two different solitonic branches in accordance with the number of structural elements contained in the unit cell of primary ladder lattice.

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